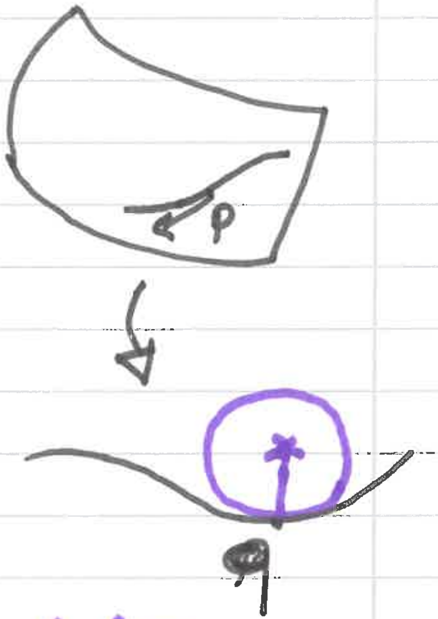
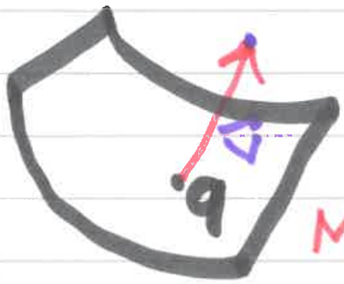
★ Manifolds in Euclidean space ★

$M \to \mathbb{R}$

$L_p = \|q - p\|^2$



1. Critical points
2. Focal points
3. Curvature



$M \subset \mathbb{R}^m$

$N = (M, \mathbb{R}^m)$

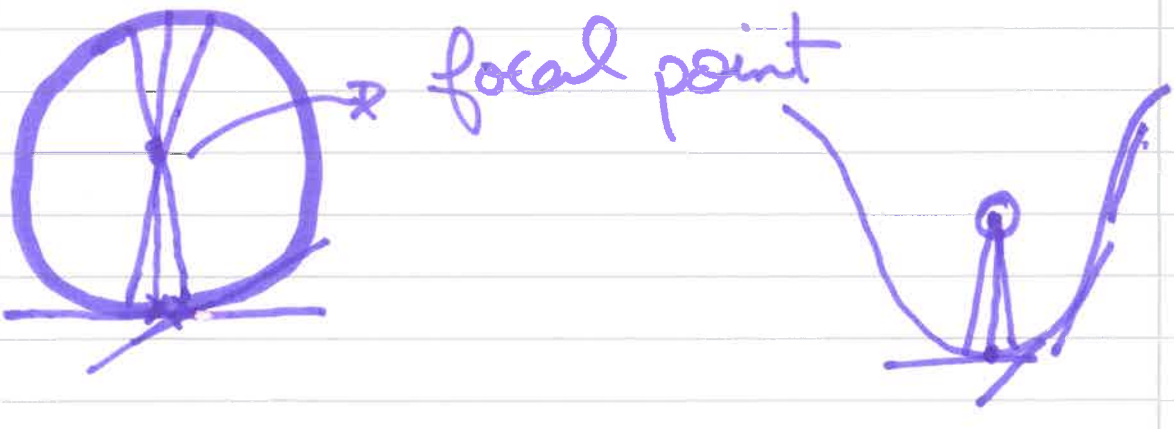
$= \left\{ (q, v) \ , \ \begin{array}{l} q \in M \\ v \perp (M, \bar{q}) \end{array} \right\}$

$E : N \longrightarrow \mathbb{R}^m$

$(q, v) \longrightarrow q + v$

$J(E)$, if $\exists \ (q, v)$ such that $J(E)$ in (q, v) is singular then

$q + V \longrightarrow$ focal point



Theorem (Sard)

M_1, M_2 are two manifolds countable bases, if we have a differentiable fct $f: M_1 \to M_2$ and f is of class C^1, then the set of critical (~~points~~ values) has a Lebesgue measure of 0 in M_1.

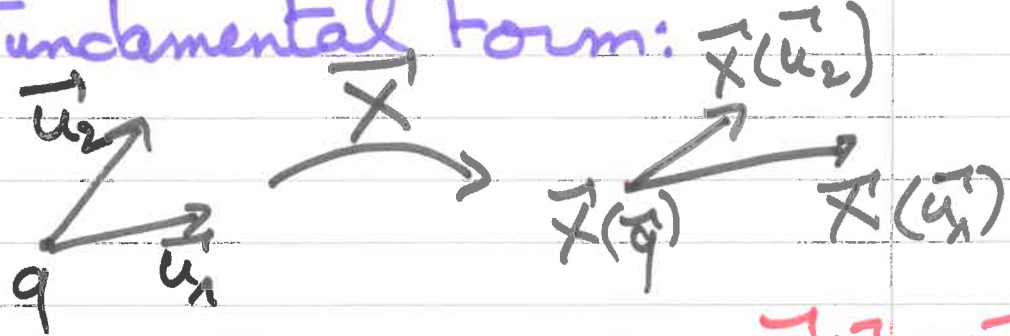
Corollary:

The set of focal points also has a measure 0.

Proof: focal points are images of critical points.

$\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k$ $\qquad$ $\vec{x}_1(u^1, \ldots, u_k) \ldots, \vec{x}_m(u^1, \ldots u_k)$

First Fundamental Form:



$$\langle \vec{x}(\vec{u}_2), \vec{x}(\vec{u}_1) \rangle = \vec{u}_2 \, \underbrace{\nabla \vec{x}(\vec{q}) \nabla \vec{x}(\vec{q})^T}_{} \, \vec{u}_1$$

$$\nabla \vec{x}(\vec{q}) = \begin{pmatrix} \frac{\partial \vec{x}}{\partial \vec{u}_1}(\vec{q}) \\ \frac{\partial \vec{x}}{\partial \vec{u}_2}(\vec{q}) \end{pmatrix}$$

$$\begin{pmatrix} \left(\frac{\partial \vec{x}}{\partial \vec{u}_1}\right)^2 & \frac{\partial \vec{x}}{\partial \vec{u}_1} \frac{\partial \vec{x}}{\partial \vec{u}_2} \\ \frac{\partial \vec{x}}{\partial \vec{u}_1} \frac{\partial \vec{x}}{\partial \vec{u}_2} & \left(\frac{\partial \vec{x}}{\partial \vec{u}_2}\right)^2 \end{pmatrix}$$

$$(\mathrm{I}) : \quad (g_{ij}) = \left(\frac{\partial \vec{x}}{\partial \vec{u}_i} \cdot \frac{\partial \vec{x}}{\partial \vec{u}_j} \right)$$

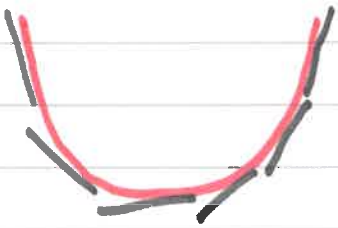
$$\left(\frac{\partial \vec{x}}{\partial \vec{u}_1}, \dots, \frac{\partial \vec{x}}{\partial \vec{u}_m} \right) \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \frac{\partial \vec{x}}{\partial \vec{u}_1}$$

$$\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} = \frac{\partial \vec{x}}{\partial \vec{u}_m}$$

Second Fundamental Form:

$$\frac{\partial^2 \vec{x}}{\partial \vec{u}_i \, \partial \vec{u}_j} = \vec{t} + \vec{n}$$

$$\vec{v} \cdot \frac{\partial^2 \vec{x}}{\partial^2 \vec{u}_i \, \partial \vec{u}_j} = \vec{v} \cdot \ell_{ij}$$



Symmetric

$(\vec{v} \cdot \ell_{ij}) \to K_1, \dots, K_n$

→ principal curvature

$K_1^{-1}, \dots, K_n^{-1}$

→ principal radii of curvature

$K = \frac{1}{r}$

Lemma: we consider the line $\ell := \vec{q} + t\vec{v}$.
The focal points at $(M, \vec{q})$ along this line are the points $\vec{q} + K_i^{-1}\vec{v}$.

$$f(p) = f(q) + \frac{1}{r}\vec{n}$$

Proof Lemma:

$$\vec{q} + K_i^{-1}\vec{V}$$

$\vec{\omega}_1, \dots, \vec{\omega}_{m-k}$, orthogonal to M

$\forall i, \forall j \neq i, \vec{\omega}_i \cdot \vec{\omega}_j = 0$

$(\vec{u}_1, \dots, \vec{u}_k)$

$(t_1, \dots, t_{m-k})$

$$E : N \longrightarrow \mathbb{R}^m$$

$$(\vec{u}_1, \dots, \vec{u}_k, t_1, \dots, t_{m-k}) \longrightarrow \vec{x}(\vec{u}_1, \dots, \vec{u}_k) + \sum_\alpha t_\alpha \vec{\omega}_\alpha(\vec{u}_1, \dots, \vec{u}_k)$$

$$\begin{cases} \dfrac{\partial E}{\partial \vec{u}_i} = \dfrac{\partial \vec{x}}{\partial \vec{u}_i} + \sum_\alpha t_\alpha \dfrac{\partial \vec{\omega}_\alpha}{\partial \vec{u}_i}(\vec{u}_1, \dots, \vec{u}_k) \\ \dfrac{\partial E}{\partial t_\beta} = \vec{\omega}_\beta \end{cases}$$

$\dfrac{\partial \vec{x}}{\partial \vec{u}_1}, \dots, \dfrac{\partial \vec{x}}{\partial \vec{u}_k}, \vec{\omega}_1, \dots, \vec{\omega}_\beta$

$$J(F) = \begin{pmatrix} \dfrac{\partial F_1}{\partial x_1}, \dots, \dfrac{\partial F_m}{\partial x_m} \\ \dfrac{\partial F_1}{\partial x_m}, \dots, \dfrac{\partial F}{\partial x_m} \end{pmatrix}$$

$$\begin{pmatrix} \left(\dfrac{\partial E}{\partial \vec{u}_i} \cdot \dfrac{\partial \vec{x}}{\partial \vec{u}_j}\right) & \left(\dfrac{\partial E}{\partial \vec{u}_i} \cdot \vec{\omega}_\beta\right) \\ \left(\dfrac{\partial E}{\partial t_\beta} \cdot \dfrac{\partial \vec{x}}{\partial u_j}\right) & \left(\dfrac{\partial E}{\partial t_\beta} \cdot \vec{\omega}_\beta\right) \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\partial \vec{x}}{\partial \vec{u}_i} \cdot \frac{\partial \vec{x}}{\partial \vec{u}_j} + \sum_\alpha t_\alpha \frac{\partial \vec{\omega}_\alpha}{\partial \vec{u}_i} \cdot \frac{\partial \vec{x}}{\partial \vec{u}_j} & \sum_\alpha t_\alpha \frac{\partial \vec{\omega}_\alpha}{\partial \vec{u}_i} \vec{\omega}_\beta \\ \vec{\omega}_\alpha \cdot \frac{\partial \vec{x}}{\partial \vec{u}_j} & \vec{\omega}_\alpha \vec{\omega}_\beta \end{pmatrix}$$

$$J_E = \begin{pmatrix} \boxed{\left(\frac{\partial \vec{x}}{\partial \vec{u}_i} \cdot \frac{\partial \vec{x}}{\partial \vec{u}_j} + \sum_\alpha t_\alpha \frac{\partial \vec{\omega}_\alpha}{\partial \vec{u}_i} \cdot \frac{\partial \vec{x}}{\partial \vec{u}_j}\right)}_{A} & \sum_\alpha t_\alpha \frac{\partial \vec{\omega}_\alpha}{\partial \vec{u}_i} \vec{\omega}_\beta \quad B \\ \begin{pmatrix} & 0 & \end{pmatrix}_{C} & \left(I_{m-k}\right) \quad D \end{pmatrix}$$

Jacobian $m \times m$

$$\det(J_E) = \det(A) \det(D - CA^{-1}B)$$
$$= \det(A)$$

$$\to \frac{\partial \vec{x}}{\partial \vec{u}_i} \cdot \vec{\omega}_\beta = 0$$

$$\Rightarrow \frac{\partial}{\partial \vec{u}_j} \left(\frac{\partial \vec{x}}{\partial \vec{u}_i} \cdot \vec{\omega}_\beta \right) = 0$$

$$\Rightarrow \frac{\partial^2 \vec{x}}{\partial \vec{u}_j \vec{u}_i} \cdot \vec{\omega}_\beta + \frac{\partial \vec{x}}{\partial \vec{u}_i} \cdot \frac{\partial \vec{\omega}_\beta}{\partial \vec{u}_j} = 0$$

$$\Rightarrow e_{ij} \cdot \vec{\omega}_\beta + \frac{\partial \vec{x}}{\partial \vec{u}_i} \cdot \frac{\partial \vec{\omega}_\beta}{\partial \vec{u}_j} = 0$$

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$$A = \left(\frac{\partial \vec{x}}{\partial u_i} \cdot \frac{\partial \vec{x}}{\partial u_j} + \sum_\alpha t_\alpha \frac{\partial \vec{\omega}_\alpha}{\partial u_i} \frac{\partial \vec{x}}{\partial u_j} \right)$$

$$= \left(g_{ij} - \sum_\alpha \underbrace{t_\alpha e_{ij} \cdot \vec{\omega}_\alpha} \right)$$

Some point $(\vec{q}, \vec{v}) \in N$, A is singular when the point $\vec{p} = \vec{q} + t\vec{v}$ is the focal point of M at $(M, \vec{q})$

$$\Big(\underbrace{g_{ij}}_{\text{eigenvalues equal to } 1} - \underbrace{t\vec{v} \cdot e_{ij}}_{\text{eigenvalues } K_1, \dots, K_m} \Big)$$

$\rightarrow$ Singular when for some point:

$$1 - tK_i = 0$$

$$t = \frac{1}{K_i}, \quad \forall\, i \in \{1, \dots, m\}, \; K_i \neq 0$$

Our focal points can be written as: $\vec{p} = \vec{q} + \frac{1}{K_i} \vec{v}$

$L_p : q \longmapsto \|p - q\|^2$

$p \in \mathbb{R}^m$

$$f(\vec{x}(\vec{u_1}, \dots, \vec{u_k})) = \|\vec{p} - \vec{x}\|^2$$
$$= \vec{p}\cdot\vec{p} - 2\vec{p}\cdot\vec{x} + \vec{x}\cdot\vec{x}$$

$$* \quad \frac{\partial f}{\partial \vec{u_i}} = 2\frac{\partial \vec{x}}{\partial \vec{u_i}}\cdot\vec{x} - 2\vec{p}\cdot\frac{\partial \vec{x}}{\partial \vec{u_i}}$$
$$= 2\frac{\partial \vec{x}}{\partial \vec{u_i}}\cdot(\vec{x} - \vec{p})$$

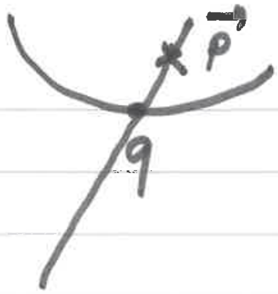
if we take $q \in M$, it's a critical point iff $(\vec{q} - \vec{p})$ is $\perp (M, \vec{q})$

$$* \quad \frac{\partial^2 f}{\partial \vec{u_i}\,\partial \vec{u_j}} = \frac{2\partial^2 \vec{x}}{\partial \vec{u_i}\,\partial \vec{u_j}}(\vec{x} - \vec{p}) + \frac{2\partial \vec{x}}{\partial \vec{u_i}}\frac{\partial \vec{x}}{\partial \vec{u_j}}$$
$$= 2 e_{ij}(\vec{x} - \vec{p}) + 2 g_{ij}$$
$$= 2(g_{ij} + (\vec{x} - \vec{p})e_{ij})$$

If we take $\vec{p} = \vec{q} + t\vec{v}$:

$$\frac{\partial^2 f}{\partial \vec{u_i}\,\partial \vec{u_j}} = 2(g_{ij} - t\vec{v}\,e_{ij})$$

$\vec{p} = \vec{q} + t\vec{v}$



if it's a focal point $\rightarrow$ $\vec{p} = \vec{q} + K_i^{-1}\vec{v}$

Lemma:

A point $\vec{q} \in M$ is a degenerate critical point iff $\vec{p} = \vec{q} + \underbrace{t}_{(K_i^{-1})}\vec{v}$ is a focal point of $(M, \vec{q})$.

Theorem:

For almost all $p \in \mathbb{R}^n$, our function

$$L_p : M \longrightarrow \mathbb{R}$$
$$q \longmapsto \|p-q\|^2$$

has no degenerate critical points.

Theorem 3.5

If f is a differentiable function on a manifold M with no degenerate critical points, and if each M^a is compact then M has the homotopy type of a CW-complex with one cell of dimension λ for each critical point of index λ.

$$\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(A)\det(D - CA^{-1}B)$$
$$= \det(A)\det(D)$$
$$= \det(A) = 0$$

critical point

$CA^{-1} + MA$

Jacobian on $\frac{\partial \vec{e}}{\partial u_i}$, $\frac{\partial \vec{e}}{\partial t^\beta}$ — we get the components in every component by projecting on them.

This is why we multiply with the linearly independent vectors $\frac{\partial \vec{x}}{\partial u_1}, \ldots \frac{\partial \vec{x}}{\partial u_k}, \vec{w}_1, \ldots \vec{w}_{n-k}$